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Gauge covariance and the gauge technique†

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Abstract. It is established that the spectral *ansätze* for longitudinal Green functions in first approximation of the gauge technique fulfil the covariance properties expected on general grounds in ultraviolet and infrared regimes, but possibly not at intermediate momenta. Explicit calculations in quantum electrodynamics confirm these statements.

1. Introduction

The gauge technique (Salam 1963, Delbourgo 1979) is a non-perturbative method of solving the coupled Green function equations of a gauge field theory, guaranteeing from the very beginning that the Ward identities are automatically respected. It has recently been pointed out by Slim (1980) that satisfaction of the Ward identities is a weaker constraint than requiring fulfilment of the entire gauge covariance relations, and he purports to demonstrate an inconsistency between the spectral functions derived in first gauge approximation (Delbourgo and West 1977a) by the technique and their complete gauge transformation properties (Zumino 1960) expected on general grounds. This demonstration is stated by Slim to be already apparent in order e^2, a puzzling comment in view of the fact that the spectral functions are known to be exact at this order if nothing else. Equally puzzling is the fact that the counterpart scalar problem (Delbourgo and Keck 1980) seems to work out consistently for all momenta. We therefore thought it worthwhile to check whether this discrepancy really exists in the context of spinor electrodynamics. Contrary to Slim, we find that the covariance property is properly satisfied to order e^2 and that it also holds at infrared and ultraviolet momentum limits. However, Slim's general conclusion is perfectly right in as much as there does occur a conflict between full gauge covariance and the specific spectral *ansätze* (for longitudinal amplitudes) at subasymptotic regimes. Though this is somewhat disappointing—for it teaches us that fulfilling the Ward identities is not everything in gauge models—it is not especially tragic either; for one thing a great deal of information is contained in the asymptotic momentum limits, where the technique does its job satisfactorily (Johnson and Zumino 1959, Baker *et al* 1964), and for another the importance of transverse corrections‡ to the amplitudes in intermediate regimes (which presumably patch up the gauge transformation properties) has always been recognised.

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‡ These give contributions of order e^4 at least to the spectral function and can be taken into account by proceeding to higher orders of gauge approximation, i.e. by solving the higher-point Green function identities with the inclusion of transverse components in lower point functions.

In the next section we have distilled the essence of the possible mismatch between the gauge technique *ansätze* and the gauge covariance properties without regard to detailed calculations. There we prove that the clash cannot arise in the various asymptotic limits but only for subasymptotic momenta. In the following section we analyse the Mellin transformed covariance relations (Slim 1980) in quantum electrodynamics to see what is implied for the asymptotic domain. Because Slim points out the discrepancy with reference to a particular solution of the technique equations for the spectral functions, we have re-examined these equations in § 4 and extracted their solutions by a completely different route from Slim. Our ultraviolet behaviours do not quite agree with Slim's; this may be due to his particular choice of Meijer function or for some other reason. But in any event we are able to exhibit a conflict between the Mellin transform of the spectral equations and the relations found in § 3; the conflict dissipates at various limits, in conformity with the general arguments of § 2.

2. Gauge covariance relations

In electrodynamics, under a gauge change of photon propagator,

$$D_{\mu\nu}(z) \to D_{\mu\nu}(z) - \partial_\mu \partial_\nu M(z), \tag{1}$$

the transformation properties of the Green functions were systematically established long ago by Zumino (1960). The first few, which chiefly concern us, read

$$S(x) \to \exp(\mathrm{i}e^2 M(x)) S(x) \tag{2a}$$

$$(S\Gamma SD)_\mu(x, y; z) \to \exp(\mathrm{i}e^2 M(x-y))(S\Gamma SD)_\mu(x, y; z) + \mathrm{i}eS(x-y)\partial^z_\mu[M(x-z) - M(y-z)] \tag{2b}$$

where S is the electron propagator and Γ is the proper three-point vertex, both renormalised. Letting δX stand for the change in the quantity X for infinitesimal M in (1), the covariance relations become

$$\delta S(x) = \mathrm{i}e^2 M(x) S(x) \tag{3a}$$

$$\delta(S\Gamma SD)_\mu(x, y; z) = \mathrm{i}e^2 M(x-y)(S\Gamma SD)_\mu(x, y; z) + \mathrm{i}eS(x-y)\partial^z_\mu[M(x-z) - M(y-z)]. \tag{3b}$$

In the gauge technique one works with the photon amputated functions $S\Gamma S$ and in terms of these the covariance property is transcribed into

$$\delta(S\Gamma_\lambda S)(x, y; z) = \mathrm{i}e^2 M(x-y)(S\Gamma_\lambda S)(x, y; z). \tag{3c}$$

Of course there is also the Ward–Takahashi identity

$$\partial^\lambda_z(S\Gamma_\lambda S)(x, y; z) = \mathrm{i}e[\delta^4(z-y)S(x-z) - \delta^4(x-z)S(z-y)] \tag{4}$$

valid in all gauges. The Schwinger–Dyson equation too is a gauge-covariant dynamical statement.

The gauge technique hinges upon finding 'solutions' of gauge identities such as (4); this necessarily only provides a longitudinal solution, since transverse amplitudes (divergenceless at the photon legs) are missed out. The transverse components naturally come in via the dynamical equations and they also enter into the covariance properties (3) expected in general. Now the basic *ansätze* for the longitudinal amplitudes are spectral weightings over Born graphs:

$$S(x)=\int \mathrm{d}W\rho(W)S(x|W) \tag{5a}$$

$$(S\Gamma''_\lambda S)(x, y; z)=\int \mathrm{d}W\rho(W)S(x-z|W)\gamma_\lambda S(z-y|W)\ldots. \tag{5b}$$

Here $S(x|W)$ stands for the *free* spinor propagator corresponding to a mass W fermion. Identities (4) and higher are automatically satisfied. However, Slim has rightly questioned if this is sufficient to guarantee the full covariance properties (3). We shall now show that it is, but only in asymptotic momentum regimes.

Turning Slim's question around, it is incumbent upon us to demonstrate that the longitudinal *ansätze* (5) alone satisfy ($3c$) given ($3a$); or equivalently that the covariance properties of longitudinal and transverse amplitudes are independent. In momentum space, since ($3a$) and ($5a$) together mean that

$$\int \mathrm{d}W\,\delta\rho(W)S(p|W)=\mathrm{i}e^2\int \mathrm{d}^4q\, S(p-q|W)\rho(W)M(q) \tag{6}$$

and because

$$(S\Gamma''_\lambda S)(p, p-k)=\int \mathrm{d}W S(p|W)\gamma_\lambda S(p-k|W)\rho(W), \tag{7}$$

where $S(p|W)\equiv(\gamma\cdot p-W)^{-1}$, we have to investigate under what conditions

$$\int \mathrm{d}W\,\delta\rho(W)S(p|W)\gamma_\lambda S(p-k|W)$$

can equal

$$\mathrm{i}e^2\int \mathrm{d}^4q\,\mathrm{d}W\rho(W)S(p-q|W)\gamma_\lambda S(p-k-q|W)M(q);$$

in other words, in what circumstances do the transverse vertices decouple?

It is easy to discern what is happening in the infrared limit, $k\to 0$ in (7). For, using (6), one immediately recognises

$$\int \mathrm{d}W\,\delta\rho(W)S(p|W)\gamma_\lambda S(p|W)=-(\partial/\partial p)^\lambda\int \mathrm{d}W\,\delta\rho(W)S(p|W)$$

$$=\mathrm{i}e^2\int \mathrm{d}^4q\,\mathrm{d}W S(p-q|W)\gamma_\lambda S(p-q|W)\rho(W)M(q),$$

establishing gauge covariance in the infrared. More generally, if k is small in relation to p the covariance property is fulfilled; this then includes the case $p\to\infty$, the *ultraviolet spinor limit.* Finally we can contemplate taking $k\to\infty$ with p fixed. Since equation (7) tends to

$$-\int \mathrm{d}W S(p|W)\gamma_\lambda\rho(W)(\gamma\cdot k)^{-1}=-S(p)\gamma_\lambda(\gamma\cdot k)^{-1}$$

it follows that

$$\delta(S\Gamma''_\lambda S)(p,p-k)\to -\delta S(p)\gamma_\lambda(\gamma\cdot k)^{-1}=-\mathrm{i}e^2\int \mathrm{d}^4q\,\delta S(p-q)M(q)\gamma_\lambda(\gamma\cdot k)^{-1}$$

$$=\lim_{k\to\infty}\mathrm{i}e^2\int \mathrm{d}^4q\,\mathrm{d}W\,S(p-q|W)\gamma_\lambda S(p-k-q|W)\rho(W)M(q).$$

So once again, this time in the *photon ultraviolet limit*, gauge covariance holds good. Our deductions break down however for p comparable to k, namely for intermediate momenta, as there is no obvious reason why, given (3a), the *ansatz* (5b) or (7) will by itself transform according to (3c). Indeed, for spinor electrodynamics as Slim has pointed out, and as we shall presently verify, it does not. Nevertheless it is worth bearing in mind that the *ansätze* are fully satisfactory† at asymptopia as well as to order e^2 (where ρ is exact), which is just the regime where most applications to date have been made.

3. Mellin transforms

Hereafter we shall stick to gauge transformations associated with $M(q)=a/q^4$, corresponding to ‡

$$\exp(\mathrm{i}e^2M(x))\to(m^2r^2)^{-a\varepsilon}\qquad \text{with } \varepsilon\equiv e^2/16\pi^2. \tag{8}$$

The transformation property of the spectral function $\rho(W;a)$ was described in an earlier paper (Delbourgo and Keck 1980) and is reproduced below. Let

$$\rho(W;a)\equiv\varepsilon(W)[W\rho_1(W^2;a)+m\rho_2(W^2;a)]$$
$$\equiv\varepsilon(W)W^{-2}[W\sigma_1(x;a)+m\sigma_2(x;a)] \tag{9}$$

where $x\equiv W^2/m^2$ is the dimensionless argument. Then§

$$\sigma_1(x;a)=\int_1^x \mathrm{d}x'\,\frac{x'}{x^{2-a\varepsilon}}\frac{(1-x'/x)^{-1+2a\varepsilon}}{2^{2a\varepsilon}\Gamma(2a\varepsilon)}F(2+a\varepsilon,a\varepsilon;2a\varepsilon;1-x'/x)\sigma_1(x':0) \tag{10a}$$

$$\sigma_2(x;a)=\int_1^x \mathrm{d}x'\,\frac{1}{x^{1-a\varepsilon}}\frac{(1-x'/x)^{-1+2a\varepsilon}}{2^{2a\varepsilon}\Gamma(2a\varepsilon)}F(1+a\varepsilon,a\varepsilon;2a\varepsilon;1-x'/x)\sigma_2(x';0) \tag{10b}$$

gives the relation between the spectral functions in gauge a with those in Landau gauge $a=0$.

† It is perhaps amusing to note that certain combinations of longitudinal amplitudes still transform correctly. Thus

$$(S\Gamma''S)(p,p-k)-(S\Gamma''S)(p+k,p)$$

satisfies the gauge covariance relation to order k—this is verified by taking a second derivative of (6) with respect to p and contracting once with k. Similarly

$$(S\Gamma''S)(p,p-k)-4(S\Gamma''S)(p+\tfrac{1}{2}k,p-\tfrac{1}{2}k)+(S\Gamma''S)(p+k,p)$$

is gauge covariant in order k^2, etc.

‡ We have normalised our propagator $S(p)$ at $p=0$ differently from Slim. It makes no essential difference to the argument. Slim's u factors merely disappear.

§ The identity $F(a,b;c;1-z^{-1})=z^aF(a,c-b;c;1-z)$ has been applied to change the argument of the hypergeometric function in Delbourgo and Keck (1980).

Next take the Mellin transforms

$$s_i(y) \equiv \int \mathrm{d}x\, x^{-y}\sigma_i(x). \tag{11}$$

Since the integration range in (10) is $1<x'<x<\infty$ one can rearrange the order of integration from $\int_1^\infty \mathrm{d}x \int_1^x \mathrm{d}x'$ to $\int_1^\infty \mathrm{d}x' \int_{x'}^\infty \mathrm{d}x$ and carry out the integrals with the aid of

$$\int_0^1 \mathrm{d}t\,(1-t)^{c-1}t^{c'-c-1}F(a,b,c;1-t) = \Gamma(c)\Gamma(c'-c)\Gamma(c'-a-b)/\Gamma(c'-a)\Gamma(c'-b)$$

to obtain

$$s_1(y;a) = \frac{\Gamma(y-a\varepsilon-1)\Gamma(y-a\varepsilon+1)}{\Gamma(y-1)\Gamma(y+1)}\frac{s_1(y-a\varepsilon;0)}{2^{2a\varepsilon}} \tag{12a}$$

$$s_2(y;a) = \frac{\Gamma(y-a\varepsilon-1)\Gamma(y-a\varepsilon)}{\Gamma(y-1)\Gamma(y)}\frac{s_2(y-a\varepsilon;0)}{2^{2a\varepsilon}}. \tag{12b}$$

An equivalent restatement is the pair of recurrence relations†

$$\frac{s_1(y;a)}{s_1(y-1;a)} = \left(1-\frac{a\varepsilon}{y}\right)\left(1-\frac{a\varepsilon}{y-2}\right)\frac{s_1(y-a\varepsilon;0)}{s_1(y-a\varepsilon-1;0)} \tag{13a}$$

$$\frac{s_2(y;a)}{s_2(y-1;a)} = \left(1-\frac{a\varepsilon}{y-1}\right)\left(1-\frac{a\varepsilon}{y-2}\right)\frac{s_2(y-a\varepsilon;0)}{s_2(y-a\varepsilon-1;0)}. \tag{13b}$$

The inverse transform, $\sigma(x) = (2\pi\mathrm{i})^{-1}\int_C \mathrm{d}y\, s(y)x^{y-1}$, informs us that if $\sigma(x)$ has the ultraviolet behaviour

$$\sigma(x) \sim x^\beta(\ln x)^\gamma \qquad \text{as } x\to\infty,$$

in Landau gauge, then $s(y;0)$ has a rightmost branch point nearest the contour C at $y=1+\beta$ with a branch cut associated with the function $(y-\beta-1)^{-\gamma-1}$. (In particular, for γ integral, $y=1+\beta$ becomes a pole of order $\gamma+1$.) Relations (12) show that in a different gauge this singularity of the Mellin transform is shifted to $\beta+1+a\varepsilon$, which remains distinct from possible poles at $y=a\varepsilon+1-n$, assuming β is non-integral. At the other extreme, if Landau's $\sigma(x)$ behaves as $(x-1)^\alpha$ near the endpoint, we deduce that $s(y;0)\sim y^{-1-\alpha}$ as $y\to\infty$ is the infrared behaviour; and consequently $s(y;a)\sim y^{-1-\alpha-2a\varepsilon}$ in any gauge. In short, infrared is connected with $y\to\infty$ and ultraviolet is governed by the y singularity nearest the left of the Mellin transform contour.

4. The gauge technique

Making the *ansatz* (7) in the Dyson equation, we arrive at the following pair of integral equations for the spectral functions (Delbourgo and West 1977a),

$$x\sigma_1(x) - \sigma_2(x) = \varepsilon\int_1^x \mathrm{d}x'(ax/x'-3)\sigma_1(x') \tag{14a}$$

$$\sigma_1(x) - \sigma_2(x) = \varepsilon\int_1^x \mathrm{d}x'(3/x'-a/x)\sigma_2(x') \tag{14b}$$

† To translate these recurrences into Slim's notation, put $y=z+2$, replace $s(y-1)$ by $\Phi(z-1)$ and change ε into λ.

for all a. In deriving equations (14) it is important to remember that transverse vertices which are crucial for renormalising $\Sigma(p, p)\sigma(p)$ in the total equation (Delbourgo 1979) have been dropped. It is relatively simple to solve equation (14) in the Landau gauge $a=0$, in the sense that well-known ${}_2F_1$ transcendental functions ensue (Delbourgo and West 1977b). The problem is much harder for $a \neq 0$, but needs to be tackled if one is interested in checking explicitly for gauge covariance. We shall follow a totally different procedure from Slim in solving for the σ_i.

Differentiate equations (14) twice with respect to x ($\mathrm{D} = \mathrm{d}/\mathrm{d}x$). Then

$$\mathrm{D}[x\mathrm{D}\sigma_1 + (1 - a\varepsilon + 3\varepsilon)\sigma_1] - a\varepsilon\sigma_1/x = \mathrm{D}^2\sigma_2 \tag{15a}$$

$$\mathrm{D}[x^2\mathrm{D}(\sigma_1 - \sigma_2) + x(a-3)\varepsilon\sigma_2] = a\varepsilon\sigma_2. \tag{15b}$$

Equation (15a) allows us to eliminate $\mathrm{D}^2\sigma_2$ from (15b) and obtain a linear relation between σ_2 and $\mathrm{D}\sigma_2$. Differentiating (15b) again and again, we end up with a fourth-order equation in σ_1,

$$\mathrm{D}^3(x^2\mathrm{D}\sigma_1) - \mathrm{D}^2(x^2\mathrm{D}^2\sigma_2) - \mathrm{D}[x(2+3\varepsilon - a\varepsilon)\mathrm{D}^2\sigma_2] - (2 - a\varepsilon + 6\varepsilon)\mathrm{D}^2\sigma_2 = 0$$

because $\mathrm{D}^2\sigma_2$ can be expressed as a second-order derivative on σ_1 by (15a). We can cast this into more conventional hypergeometric form by using the differential operator $\theta = x\,\mathrm{d}/\mathrm{d}x$ in place of D. After a little work one obtains

$$\{\theta^2(\theta^2-1) - x[\theta^2 + (1 - a\varepsilon + 3\varepsilon)\theta - a\varepsilon][\theta^2 + (3 - a\varepsilon + 3\varepsilon)\theta + 2 - a\varepsilon + 6\varepsilon]\}\sigma_1 = 0. \tag{16a}$$

Similarly one finds that σ_2/x obeys the equation

$$\{\theta^2(\theta+1)^2 - x[\theta^2 + (3 - a\varepsilon + 3\varepsilon)\theta + 2 - 2a\varepsilon + 3\varepsilon] \times[\theta^2 + (3 - a\varepsilon + 3\varepsilon)\theta + 2 - a\varepsilon + 6\varepsilon]\}\sigma_2/x = 0. \tag{16b}$$

Now the hypergeometric function

$${}_4F_3(a_1, a_2, a_3, a_4; c_1, c_2, c_3; x) = \sum_n \frac{(a_1)_n(a_2)_n(a_3)_n(a_4)_n}{(c_1)_n(c_2)_n(c_3)_n n!}x^n \tag{17a}$$

formally satisfies the differential equation

$$[\theta(\theta + c_1 - 1)(\theta + c_2 - 1)(\theta + c_3 - 1) - x(\theta + a_1)(\theta + a_2)(\theta + a_3)(\theta + a_4)]F = 0. \tag{17b}$$

Thus we can recognise σ_1 as a ${}_4F_3$ function possessing the parameters

$$\begin{gathered} c_1 = 0, \qquad c_2 = 1, \qquad c_3 = 2, \\ a_{1,2} = \tfrac{1}{2}(1 - a\varepsilon + 3\varepsilon) \mp \tfrac{1}{2}[1 + 2a\varepsilon + 6\varepsilon + (a-3)^2\varepsilon^2]^{1/2} \\ a_{3,4} = \tfrac{1}{2}(3 - a\varepsilon + 3\varepsilon) \mp \tfrac{1}{2}[1 - 2a\varepsilon - 6\varepsilon + (a-3)^2\varepsilon^2]^{1/2}. \end{gathered} \tag{18}$$

Since the series (17a) is strictly undefined for $c_1 = 0$ we have to reinterpret the function by a renormalisation of $1/\Gamma(c_1)$, just as one does in the ${}_2F_1$ case. This yields the hypergeometric functions of *type*

$$\begin{aligned} \rho_1 &\leftrightarrow {}_4F_3(a_1+1, a_2+1, a_3+1, a_4+1; 2, 2, 3; x) \\ \rho_2 &\leftrightarrow {}_4F_3(a_1+1, a_2+1, a_3, a_4; 1, 2, 2; x) \end{aligned} \tag{19a}$$

or, by examination of the associated equations, Meijer functions of *type*

$$\rho_1 \leftrightarrow G_{44}^{mn}\left((-1)^{m+n}x \middle| \begin{matrix} -a_1, -a_2, -a_3, -a_4 \\ 0, -1, -1, -2 \end{matrix}\right)$$

$$\rho_2 \leftrightarrow G_{44}^{mn}\left((-1)^{m+n}x \middle| \begin{matrix} -a_1, -a_2, 1-a_3, 1-a_4 \\ 0, 0, -1, -1 \end{matrix}\right). \qquad (19b)$$

Of course that is only the character of the function: there are several linearly independent solutions of the equation and it is essential to determine the right combination which satisfies the original integral equations (14). The correct solution needs to reproduce first-order perturbation theory, and when $a = 0$ it must reduce to the known answer (Delbourgo and West 1977b) for the Landau gauge; namely, for σ_1 say, $(x-1)^{-1-6\varepsilon}{}_2F_1(-3\varepsilon, -3\varepsilon; -6\varepsilon; 1-x)$, *not* the solution ${}_2F_1(1+3\varepsilon, 1+3\varepsilon; 1; x)$.

By contrast with the normal ${}_2F_1$ or G_{22}, there is a dearth of literature on the fundamental system of solutions for ${}_4F_3$ or G_{44} in the neighbourhood of $x = 1$; the only relevant mathematical reference we have been able to trace which treats this point in any detail is an article by Norlund (1955). Abstracting his analysis to our circumstances, we are interested in the solution which behaves as

$$(x-1)^{-1+2(a-3)\varepsilon} = (x-1)^{\Sigma c - \Sigma a}$$

near $x = 1$, and are led to picking Norlund's function ξ. For ρ_1 this is

$$\xi = 2(a-3)\varepsilon(1-x)^{-1+2(a-3)\varepsilon} \sum_n^{\infty} \frac{C_n}{(2(a-3)\varepsilon)_n}(1-x)^n. \qquad (20)$$

It is not clear to us that equation (20) is precisely the same as Slim's solution G_{44}^{04}. Both our answer and Slim's certainly reduce to the correct Landau gauge solutions above, but they seem to disagree in the ultraviolet limit. For us, just as for Slim,

$$\sigma_1 \text{ and } \sigma_2 \to (x-1)^{-1+2(a-3)\varepsilon} \qquad \text{as } x \to 1; \qquad (21)$$

but unlike Slim, we contend that ultraviolet self-consistency of (14) is achieved by

$$\sigma_1 \sim x^{-a_2}, \qquad \sigma_2 \sim x^{1-a_3} \qquad \text{as } x \to \infty. \qquad (22)$$

Note that when the gauge parameter $a \to 0-$, the next to leading behaviour in σ_1, x^{-a_3}, competes with (22) to provide the ultraviolet behaviour

$$\sigma_1 \sim x^{-1-3\varepsilon}(\ln x), \qquad \sigma_2 \sim x^{-3\varepsilon},$$

expected in the Landau gauge—which then leads one to

$$S(p) \sim (\gamma p)^{-1} + (3\varepsilon m)^{-1}(-p^2/m^2)^{-1-3\varepsilon}, \qquad (23)$$

in perfect agreement with the self-consistent asymptotic behaviour found by Baker *et al* (1964). The absence of a logarithm in Slim's quoted behaviour† may be due to his keeping away from $a = 0$. However it ought to be retrieved when $a = 0$ since one is then dealing with a degenerate situation.

Whatever the rights and wrongs of the proposed solutions, one may readily uncover an inconsistency between equations (10) and (14). Take the Mellin transform of (14).

† It also makes us slightly suspicious of his G_{44}^{04} choice, although we cannot be categorical about this. To our mind G_{44}^{13} looks a more plausible solution for ρ_1 and ρ_2.

This gives the gauge technique recurrences

$$s_1(y-1)\left[1+\varepsilon\left(\frac{3}{y-1}-\frac{a}{y-2}\right)\right]=s_2(y) \tag{24a}$$

$$s_2(y)\left[1+\varepsilon\left(\frac{3}{y-1}-\frac{a}{y}\right)\right]=s_1(y) \tag{24b}$$

for all a. More particularly the ratios

$$\frac{s_1(y;a)}{s_1(y-1;a)}=\left[1+\varepsilon\left(\frac{3}{y-1}-\frac{a}{y}\right)\right]\left[1+\varepsilon\left(\frac{3}{y-1}-\frac{a}{y-2}\right)\right] \tag{25a}$$

$$\frac{s_2(y;a)}{s_2(y-1;a)}=\left[1+\varepsilon\left(\frac{3}{y-1}-\frac{a}{y-2}\right)\right]\left[1+\varepsilon\left(\frac{3}{y-2}-\frac{a}{y-1}\right)\right] \tag{25b}$$

allow us to evaluate

$$\frac{s_1(y-a\varepsilon;0)}{s_1(y-a\varepsilon-1;0)}=\left(1+\frac{3\varepsilon}{y-a\varepsilon-1}\right)^2 \tag{26a}$$

$$\frac{s_2(y-a\varepsilon;0)}{s_2(y-a\varepsilon-1;0)}=\left(1+\frac{3\varepsilon}{y-a\varepsilon-1}\right)\left(1+\frac{3\varepsilon}{y-a\varepsilon-2}\right). \tag{26b}$$

When substituted back into the covariance recurrences (13) one deduces

$$\frac{s_1(y;a)}{s_1(y-1;a)}=\left(1-\frac{a\varepsilon}{y}\right)\left(1-\frac{a\varepsilon}{y-2}\right)\left(1+\frac{3\varepsilon}{y-a\varepsilon-1}\right)^2$$

$$\frac{s_2(y;a)}{s_2(y-1;a)}=\left(1-\frac{a\varepsilon}{y}\right)\left(1-\frac{a\varepsilon}{y-2}\right)\left(1+\frac{3\varepsilon}{y-a\varepsilon-2}\right)\left(1+\frac{3\varepsilon}{y-a\varepsilon-1}\right)$$

which can be seen to conflict with (25). Observe though that the mismatch disappears to order ε and also for large y (corresponding to the infrared limit), in support of the arguments of § 2. Turning to the ultraviolet limit obtained by the technique, we know from (23) that $s_1(y;0)$ has a rightmost double pole at $y=-3\varepsilon$ and $s_2(y;0)$ has a rightmost single pole at $1-3\varepsilon$. Gauge covariance (12) would then tell us, for general a, that $s_1(y;a)$ has a single pole at $y=1+a\varepsilon$ as well as a double pole at $y=(a-3)\varepsilon$, while $s_2(y;a)$ has leading single poles at $1+(a-3)\varepsilon$ and $1+a\varepsilon$. But (25), derived without reference to gauge covariance, shows that the ratio $s_1(y+1)/s_1(y)$ has zeros at

$$\begin{aligned} y=1-a_1, \qquad & 1-a_2, \qquad & 1-a_3, \qquad & 1-a_4 \\ =1+a\varepsilon-3a\varepsilon^2, \qquad & -3\varepsilon+3a\varepsilon^2, \qquad & -3\varepsilon-3a\varepsilon^2, \qquad & -1+a\varepsilon+3a\varepsilon^2 \end{aligned}$$

to order ε^2.

The position of the rightmost pole is thus confirmed to order ε, but the nearby double pole is not found; instead there are two finely separated single poles at a slightly different location from $(a-3)\varepsilon$. Analogous statements apply to σ_2 (which provides the most significant correction to the bare propagator): the two leading poles are not quite at $1+a\varepsilon$ and $1+(a-3)\varepsilon$ but at $1-a_1\approx 1+a\varepsilon$ and $2-a_3\approx 1-3\varepsilon$.

Summarising, we assert that the gauge technique respects the full gauge covariance (not purely the satisfaction of the gauge identities) at asymptotic momenta, but conflicts mildly with it at intermediate values of momentum. The blame falls fairly and squarely on transverse vertices, and once these are incorporated into the technique by going to higher orders of gauge approximation, the conflict between the two should be resolved. In spite of the complaint, we believe that the longitudinal Green functions derived by the technique do a useful job of interpolating between infrared (Johnson and Zumino 1959) and ultraviolet regions (Baker *et al* 1964), where their behaviours are eminently respectable.

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